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The work done in this project provides new mathematical tools necessary for the description of the qualitative behavior of composite materials. These tools are used to develop numerical algorithms for the design of composite structures. The first part of this project provides new analytical results necessary for understanding the macroscopic effects of field concentrations arising from the composite microgrometry. These results form the basis of new a numerical method for the design of elastic composites subject to stress constraints. The numerical methods provide rigorous guidance for the design of reinforced structures for maximum specific stiffness and strength. Similar results are obtained for multi phase dielectric materials. A numerical method is developed to solve design problems requiring the control of the electric field in composite dielectrics. The second part of the project predicts novel physical phenomena for reinforced materials in the presence of imperfect bonding. These results are unlike phenomena seen in more traditional treatments where the interface between constituents is treated as a "perfect bond". The difference is that reinforcement size effects are predicted. New results are found for imperfectly bonded nonlinear dielectric materials and reinforcement problems in linear elasticity. These results provide guidance for the robust design of particle and fiber reinforced elastic composite structures in the presence of imperfect bonding.

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Final Project Report: Characterization and Design of Electromagnetic, Chemical and Thermal Transport Processes for Multi-Phase Systems, AFOSR Grant F49620-99-1-0009

Robert P. Lipton.

Abstract

The work done in this project provides new mathematical tools necessary for the description of the qualitative behavior of composite materials. These tools are used to develop numerical algorithms for the design of composite structures. The first part of this project provides new analytical results necessary for understanding the macroscopic effects of field concentrations arising from the composite microgeometry. These results form the basis of new a numerical method for the design of elastic composites subject to stress constraints. The numerical methods provide rigorous guidance for the design of reinforced structures for maximum specific stiffness and strength. Similar results are obtained for multi phase dielectric materials. A numerical method is developed to solve design problems requiring the control of the electric field in composite dielectrics.

The second part of the project predicts novel physical phenomena for reinforced materials in the presence of imperfect bonding. These results are unlike phenomena seen in more traditional treatments where the interface between constituents is treated as a "perfect bond". The difference is that reinforcement size effects are predicted. New results are found for imperfectly bonded nonlinear dielectric materials and reinforcement problems in linear elasticity. These results provide guidance for the robust design of particle and fiber reinforced elastic composite structures in the presence of imperfect bonding.

Contents

1 Research overview.

During the last few years several technologically important applications have benefited from the use of functionally graded composite materials, see Glasser (1997) and Koizumi (1997). In many applications there is a separation of scales and the discrete entities forming up the microstructure exist on scales significantly smaller than the characteristic length scale of the loading. Under this hypothesis functionally graded materials are modeled using effective thermophysical properties that depend upon features of the underlying microgeometry. The effective thermophysical properties are given by effective constitutive laws relating average flux to average gradient, see Markworth, et. al. (1995).

The primary problem of design of a functionally graded material is the determination of the optimal spatial dependence for the composition. This type of problem has generated much interest in the engineering community and is the topic of a rapidly developing literature, see Markworth, et. al. (1995), and Ootao, et. al. (2000). For many objective functions this type of problem has also received significant attention from both the applied mathematics and structural optimization communities in the 1980s and 1990s under the headings of the homogenization method for topology optimization and structural optimization, see Bendose and Kikuchi (1988), Allaire and Kohn (1993), Lurie and Cherkaev (1986), Murat and Tatar (1985) and Cheng and Olhoff (1981). Homogenization methods applied to the design of composites for optimal structural performance can be found in the works of Sigmund and Torquato (1997), Fujii et. al. (2001) and Diaz and Lipton (2000). In all of these works the problem of determining the optimal spatial dependence for the composition is obtained through the use of effective constitutive relations.

Motivated by the applications, this project treats the problem of optimizing structural or dielectric properties subject to constraints on field quantities. These quantities include the local stress for elastic composites or the electric field for dielectric composites. In many applications it is of central importance to control fields inside composite structural components. Regions containing large field gradients are most likely the first to exhibit failure during service. For this type of problem the macroscopic effect of field concentrations that arise from the microstructure need to be accounted for. The concept of an effective constitutive law is by its self not sufficient to capture the effect of these concentrations. This requires new modeling beyond the notion of effective thermophysical properties. The objective of

this project is to point out a new class of macroscopic tensors relevant to the modeling of microscopic field concentrations and to present a methodology for the numerical design of functionally graded materials in the presence of constraints on field gradients.

The second part of this project is concerned with modeling imperfectly bonded particle and fiber reinforced composites. During the previous period of AFOSR support this investigator developed a methodology to assess the effectiveness of reinforcement fibers or particles on the overall transport properties of composite materials. The approach has been successful in predicting the effect of particle size and shape on the enhancement of structural and thermal transport properties in the context of electric contact resistance, coupled heat and mass transport on the interface, highly conducting interfaces, and problems of torsional rigidity with imperfectly bonded fiber reinforcements.

The work done here extends these techniques to imperfectly bonded nonlinear dielectrics, and to reinforcement problems in the context of two dimensional elastic systems.

2 Grant sponsored projects completed.

Homogenization of field fluctuations and stress constrained design of functionally graded engineering materials.

The use of functionally graded materials is rapidly expanding and now includes structural, bio mechanical, and energy conversion applications. Functionally graded materials allow the designer to tailor the material's microstructure in order to enhance structural performance.

During the course of this project new homogenization results have been obtained that describe the effect of microscopic field concentrations on the macroscopic field intensity. It has been shown that these new quantities facilitate new numerical methods for the design of functionally graded materials for maximum strength and stiffness.

Functionally graded materials (FGMs) are characterized by microstructures that are spatially variable on the macroscale. To fix ideas we consider a structural element made from a FGM consisting of elastic reinforcement fibers with stiffness tensor A_1 embedded in a more compliant elastic matrix with stiffness tensor A_2. The characteristic length scale of the microgeometry relative to the size of the structure is denoted by ε. In many applications

ε is taken to be small, i.e., $\varepsilon << 1$. For a body load f and boundary traction g the stress σ satisfies $-\text{div}\,\sigma = f$ and $\sigma_n = g$. The stress σ is written as $\sigma = \sigma^M + \sigma_\varepsilon^m$. Here σ^M is the macrostress and the macroscopic constitutive law is $\sigma^M = C^E e^M$, where e^M is the average strain and C^E is the effective elastic tensor. The macrostress satisfies $-\text{div}\,\sigma^M = f$ and $\sigma_n^M = g$. The microstress σ_ε^m depends on ε and captures the interaction between the microstructure and σ^M.

We place a constraint on the stress $\sigma^M + \sigma_\varepsilon^m$ and for a prescribed load case, the design problem is to find the fiber distribution that renders the structural element the stiffest. We tackle the design problem for situations in which an extremely large number of fibers are used (e.g., thousands of fibers) and $\varepsilon << 1$. Design problems of this sort are too large for numerical solution using existing computational methods. Instead we look for an approximation to the problem that is computationally tractable and gives nearly optimal designs for the original problem. It has been shown by this investigator that such an approximation is found by taking the $\varepsilon = 0$ limit of the original. However the correct identification of the $\varepsilon = 0$ limit design problem is delicate. In fact when considering stress constrained problems the traditional micromechanical approach to the constitutive modeling of discreet systems fails to give the complete picture. In this limit, an accurate modeling of FGMs requires that the stress constraint include the effects of the microstress. The stress constraint can be local or of integral type and is written symbolically as $C(\sigma^M + \sigma_\varepsilon^m) \leq K$. Recent work obtained during the course of this project shows that the stress constraint does not simply reduce to $C(\sigma^M) \leq K$ in the $\varepsilon = 0$ limit. For a constraint of mean square type, i.e., $\int |\sigma^M + \sigma_\varepsilon^m|^2 dx \leq K$, this investigator has shown that the homogenized constraint is given by

$$\lim_{\varepsilon \to 0} \int |\sigma^M + \sigma_\varepsilon^m|^2 dx = \int |\sigma^M|^2 dx + \int Q\sigma^M \cdot \sigma^M \, dx \leq K \tag{1}$$

where Q is a new type of effective property and is given by

$$\mathrm{Q}(\mathbf{x}) = \left\{ \sum_{\mathrm{i}=1}^{\mathbf{2}} (\mathbf{S}^{\mathbf{E}}(\mathbf{x})\mathrm{A}_{\mathrm{i}}^{2} \vec{\nabla}^{\mathrm{i}} \mathrm{C}^{\mathbf{E}}(\mathbf{x}) \mathbf{S}^{\mathbf{E}}(\underline{\mathrm{A}}, \mathbf{x}) \right\} - \mathbf{I} \tag{2}$$

Here $\mathbf{S}^{\mathbf{E}}(\mathbf{x}) = (\mathrm{C}^{\mathbf{E}}(\mathbf{x}))^{-1}$ is the effective compliance and $\vec{\nabla}^i \mathrm{C}^{\mathbf{E}}$ gives the change of the effective elasticity due to a change in the stiffness of the i^{th} component material. The convergence of the quantity $|\sigma_\varepsilon^m|^2$ is given by

$$|\sigma_\varepsilon^m|^2 \to \mathrm{Q}(\mathbf{x})\sigma^{\mathbf{M}}(\mathbf{x}) : \sigma^{\mathbf{M}}(\mathbf{x}) \tag{3}$$

in the sense of distributions and one has the estimate

$$\mathrm{Q}(\mathbf{x})\sigma^{\mathrm{M}}(\mathbf{x}) : \sigma^{\mathrm{M}}(\mathbf{x}) \leq \liminf_{\varepsilon \to 0} \sup_{\mathbf{x}\,\mathrm{in}\,\Omega} |\sigma_{\varepsilon}^{\mathrm{m}}(\mathbf{x})|^2 \tag{4}$$

for each point $\mathbf{x}$ in the domain. It is evident from (1) that new effective properties beyond C^E are required when considering the $\varepsilon = 0$ limit of stress constrained problems.

The design variables for the $\varepsilon = 0$ design problem are obtained by exchanging the original design variables with averaged design variables such as the local density of fibers. To illustrate the basic ideas we suppose that the fiber microstructure consists of a locally periodic arrangement of long parallel fibers with circular cross section. The fiber diameters are allowed to change over the macroscopic dimensions. For this case it has been shown that the design variable for the (homogenized) $\varepsilon = 0$ design problem is the local density of fibers θ. The formulas for the effective tensors $Q = Q(\theta)$ and $C^E = C^E(\theta)$ are determined numerically through the solution of suitable microscopic problems. The resulting homogenized optimal design problem is one of maximization of the structural stiffness over θ subject to the homogenized stress constraint given by (1). The design problem is solved numerically using the method of steepest decent together with the finite element method. Numerical results have been obtained for the case of torsional loading for functionally graded fiber reinforced rods.

The results reported in this project will appear in the following journals:

1. R. Lipton, Design of functionally graded composite structures in the presence of stress constraints, *International Journal of Solids and Structures.* To appear in 2002.

2. R. Lipton, Relaxation through homogenization for optimal design problems with gradient constraints, *Journal of Optimization Theory and Applications.* To appear in 2002.

Characterization of electric-field fluctuations in random composites.

The theory of composite materials has for the most part focused on the characterization of effective transport properties that relate average flux fields to average gradient fields. In the context of heterogeneous dielectric materials this type of effective property is known as the *effective dielectric*

constant. The effective dielectric constant ε^e gives the linear relation between the average electric field $\overline{\mathbf{E}}$ and the average electric displacement $\overline{\mathbf{D}}$, i.e.,

$$\overline{\mathbf{D}} = \varepsilon^e \overline{\mathbf{E}}.$$

However, it should be pointed out, that a heterogeneous dielectric material exhibits a hierarchy of "effective properties," beyond ε^e. The first effective property in this hierarchy is the covariance tensor σ. This effective property delivers the mean square fluctuation of the electric field about its average value $\overline{\mathbf{E}}$. To make this precise, the local electric field in the composite sample Ω is denoted by $\mathbf{E}(\mathbf{x})$. The local electric field is related to the local electric displacement $\mathbf{D}(\mathbf{x})$ by $\mathbf{D}(\mathbf{x}) = \varepsilon(x)\mathbf{E}(\mathbf{x})$, where $\varepsilon(x)$ is the local dielectric constant and $\operatorname{div}\mathbf{D}(\mathbf{x}) = 0$. Then σ is defined by

$$\sigma\bar{\mathbf{E}} \cdot \bar{\mathbf{E}} = \frac{1}{|\Omega|}\int_\Omega |\mathbf{E}(\mathbf{x}) - \bar{\mathbf{E}}|^2 \, d\mathbf{x}.$$

Higher order "effective properties," are defined through higher moments of the local electric field fluctuation $\mathbf{E}(\mathbf{x}) - \bar{\mathbf{E}}$, i.e.,

$$\frac{1}{|\Omega|}\int_\Omega |\mathbf{E}(\mathbf{x}) - \bar{\mathbf{E}}|^p \, d\mathbf{x}, \; p > 2.$$

It is evident that σ and ε^e are correlated as they are generated from one and the same composite sample. For two phase composites made up of dielectric constants ε_1 and ε_2, it is shown that the relation between the effective dielectric constant and the covariance tensor is given by

$$\sigma = (\varepsilon^e - \varepsilon_1 I)/\varepsilon_1 - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1}\partial_{\varepsilon_2}\varepsilon^e(\varepsilon_1, \varepsilon_2).$$

Here we have written $\varepsilon^e = \varepsilon^e(\varepsilon_1, \varepsilon_2)$ to emphasize its dependence on the dielectric constants of the constituent materials. The second term on the right hand side is the derivative of the effective dielectric constant with respect to ε_2.

Higher order effective properties provide information on the variation of the local electric field that is not captured by the effective dielectric constant. The higher order effective properties reveal the presence of regions of high field intensity inside the composite. These regions are most often the first to suffer dielectric breakdown during service.

Unfortunately, the higher order effective properties can not be obtained directly through simple boundary measurements. While the effective dielectric constant can be easily measured by subjecting a composite sample

to a uniform electric field and measuring the current passing through the boundary of the sample. In this project we have developed bounds on the covariance tensor of the electric field that are given in terms of the effective dielectric constant. This allows one to obtain estimates for field concentrations within the composite sample from simple boundary measurements. We exhibit microstructures for which these bounds are optimal. These bounds are used to recover bounds on the covariance tensor when only the phase volume fractions and the two-point correlation function are available. For isotropic composites we obtain a lower bound that is the most restrictive one in terms the volume fraction occupied by each phase. Lastly, we provide tight upper and lower bounds on the mean square field fluctuation when only the volume fractions of the two dielectric materials are known. Denoting the volume fraction of the ε_2 dielectric by θ_2 the bounds are given by,

$$0 \leq \sigma_{ij}\bar{E}_i\bar{E}_j \leq U(\theta_2, \bar{\mathbf{E}}), \tag{5}$$

where $U(\theta_2, \bar{\mathbf{E}})$ depends upon the contrast $\lambda = \varepsilon_2/\varepsilon_1$ and is given by

$$U(\theta_2, \bar{\mathbf{E}}) = \begin{cases} (\theta_2 f(h))\,|\bar{\mathbf{E}}|^2, & if\ \ \theta_2 \leq\ 1-h, \\ (\theta_2 f(1-\theta_2))\,|\bar{\mathbf{E}}|^2 & if\ \ \theta_2 \geq\ 1-h. \end{cases}$$

Here $h = 1/(\lambda - 1)$ and the function $f(z)$ is defined by

$$f(z) = \frac{z}{(h+z)^2}.$$

These bounds are the best one can find on the covariance knowing only the volume fractions of the component phases. They are the analog of the well known harmonic mean arithmetic mean bounds for the effective conductivity. Extremal sequences of configurations that attain the bounds are shown to be given by the well known finite rank laminate microstructures. The lower bound is attained by laminates of the first rank with layers oriented parallel to the applied field. On the other hand the upper bound is saturated by laminates of first or second rank depending on the magnitude of the contrast. The method for obtaining bounds developed in this project relies on the explicit computation of the convex hull of a curve. The results reported here have appeared in the following journals:

1. R. Lipton, Optimal inequalities for gradients of solutions of elliptic equations occurring in two-phase heat conductors, *SIAM J. Mathematical Analysis*, 32, 1081–1093 (2001).

2. R. Lipton, Optimal bounds on field fluctuations for random composites, *J. Applied Physics*, 88, 4287–4923 (2000).

3. R. Lipton, Optimal bounds on field fluctuations for random composites: Three dimensional Problems, *J. Applied Physics*, 89, 1371–1376 (2001).

Optimal containment and control of dc electric fields and electric displacement fields.

Over the last 25 years several methods have been developed for the design of composites with minimum energy dissipation for given source distributions and boundary conditions. These design problems appear in many contexts and include the use of dielectric and elastic composites. However, many practical problems require that field quantities be controlled directly. Unfortunately none of the computational strategies developed for minimum energy dissipation can be used. The basic reason is that the objective functions appearing in problems where the fields need to be controlled directly are not continuous with respect to G-convergence. Recently A. Velo and myself have been able to circumvent this obstacle by computing the G-continuous extension of a class functionals used in the direct control of field quantities through the use of layered microstructures.

Based upon this extension a computational methodology has been developed for the design of functionally graded composites used for the containment and control of dc fields. We start with a two-phase dielectric material occupying the design domain Ω. A resource constraint is placed on the amount of the better dielectric that can be used in any admissible design. It is assumed that a prescribed charge density has been distributed inside the design domain. It is desired that in some sub-region D of Ω that the electric field have an intensity and direction specified by a prescribed target field $\overline{\mathbf{E}}$. The goal is to find the configuration of the two dielectrics so that the actual electric field is as close as possible, in the mean square norm, to the target electric field. In this project we have introduced a tractable method for the numerical computation of minimizing sequences of configurations of the two dielectrics. The configurations are associated with materials with graded dielectric properties that may exhibit a fine scale structure composed of layers of the two dielectrics. The computational results provide rules of thumb for the design of graded dielectric materials. These results are reported in:

1. R. Lipton and A. Velo, Optimal design of gradient fields with appli-

cations to electrostatics, in *Nonlinear Partial Differential Equations and Their Applications, College de France Seminare* Series in Applied Mathematics, edited by P.G. Ciarlet et P.-L. Lions. Elsevier-Gauthier-Villars , to appear in 2002.

The effect of the interface on the dc transport properties of nonlinear composite materials.

Nonlinear dc electric conductivity is a property intrinsic to many ceramic materials. More generally, nonlinear inhomogeneous materials are pervasive, appearing in applications ranging from transient voltage protection to the selective absorption of solar energy. This work investigates the effect of the interface on the overall dc electric properties of nonlinear composite conductors. A composite consisting of a monodisperse suspension of spheres embedded in a matrix is considered. It is supposed that the interface separating the spheres and matrix is highly conducting. A critical applied voltage is found for which the electric potential inside the sample is the same as for a sample of identical shape containing no spheres whatsoever. At the critical voltage, the overall electric current passing through the sample is the same as in a homogeneous conductor. On the other hand, in the presence of an electric contact resistance at the interface, we show that there is a critical applied dc current density for which the current density inside the sample is the same as for a sample containing no spheres. The overall electric field in the sample corresponds to that associated with a homogeneous conductor made from matrix material. These effects are shown to be independent of the location of the spheres within the sample. Moreover, this effect is independent of the concentration of spheres in the sample even beyond the onset of interface percolation.

It is demonstrated from first principles that this phenomenon occurs for a wide range of nonlinear constitutive behavior. Indeed, it is shown that this phenomenon can occur when the potential energy density of each phase is a convex function of the magnitude of the electric field. This requirement naturally includes local constitutive relations associated with nonlinear behavior of the form

$$\mathrm{j} = \gamma \, |\mathrm{E}|^{t} \, \mathrm{E}$$

where j is the local current density and E is the local electric field. Here the nonlinear susceptibility γ takes different values inside each phase.

This work is joint work with David Talbot and has appeared in

1. R. Lipton and D.R.S. Talbot Journal of Applied Physics, **86**, August 1, 1999, pp. 1480–1487.

Effect of interfacial bonding on fiber reinforced shafts subject to antiplane shear.

Fiber reinforced materials are often the materials of choice for structural components appearing in aerospace and infrastructure applications. The fibers can be perfectly bonded to the matrix or bonded to the matrix with a bond that is stiffer than either the fiber or matrix phase. Over time several of the fibers may become imperfectly bonded to the matrix. In this project we consider a shaft reinforced with long fibers subjected to antiplane shear. We outline conditions for which the effects of the interface overcome the elastic properties of the fiber reinforcement. We apply a new geometric criteria that indicates when the imperfect bond compromises the elastic properties of the reinforcement fiber. The criteria is given in terms of the bond stiffness and the *surface traction to bulk stress quotient* of the fiber cross section. This criteria was introduced by this investigator during the previous AFOSR funding period. This criteria is used to provide new rigorous rules of thumb for selecting the fiber size so that the stiffness of the structural component is greater than the unreinforced matrix.

On the other hand in the applications it is anticipated that one only has statistical data on the type of bonds and bond stiffness associated with a fiber reinforced structure. We present new mathematically rigorous guidelines for the design of the overall stiffness of fiber reinforced shafts when only partial statistical information is available. These results are applied to obtain new size effects for distributions of fibers with circular cross sections. These results are consequences of new upper and lower bounds on the effective compliance tensor associated with antiplane shear loading.

These results are applied to the design of a steel reinforced concrete column subject to antiplane shear. Given that a certain percentage of the reinforcement rods are imperfectly bonded we are able to predict a range of rod diameters for which no increase in the overall shear stiffness is possible. The utility of the methods presented in this project lie in the fact that simple but very general design rules are obtained without the use of intensive computation. Naturally more detailed relationships between fiber geometry and overall structural properties will require significant numerical effort.

This work has appeared in the following journals.

1. R. Lipton, Effect of interfacial bonding on fiber reinforced shafts sub-

ject to anti plane shear, *International Journal of Solids and Structures*, 38, 369–387 (2001).

2. R. Lipton, Effect of interfacial bonding on fiber reinforced materials. In the Proceedings of the 13th ASCE Engineering Mechanics Division Specialty Conference, Johns Hopkins University, Baltimore, June 13-16 1999. (CD-ROM)

Reinforcement of elastic structures in the presence of imperfect bonding.

In this work we consider the two dimensional problem of plane strain in the presence of imperfectly bonded elastic reinforcements. The approach taken here extends the methodology developed by this investigator for scalar problems to treat reinforcement problems described by the system of linear elasticity. A new geometric criterion on the shape and size of the elastic reinforcement is found that determines when the effects of imperfect bonding overcome the benefits of the reinforcement. The criterion is given in terms of a new type of eigenvalue problem posed on the surface of the reinforcement.

Both the matrix and reinforcement are assumed to be made from isotropic elastic materials. The Lame constants for the reinforcement and matrix phases are given by μ_r, λ_r and μ_m, λ_m respectively. Here the reinforcement is stiffer than the matrix. This condition is equivalent to the requirement that $\kappa_r > \kappa_m$ and $\mu_r > \mu_m$, where κ_r and κ_m are the plane strain bulk moduli of the reinforcement and matrix respectively, i.e., $\kappa_r = \mu_r + \lambda_r$ and $\kappa_m = \mu_m + \lambda_m$.

On the other hand the interface separating the reinforcement from the matrix is imperfect. This is associated with a discontinuity in the displacement across the interface. The constitutive model for the interface is taken to be of the spring layer type. This model represents a first order approximation to more general nonlinear interface models, e.g., the yielding interface models or shear lag models. For the spring layer model the jump in displacements is proportional to the traction at the interface. The constant of proportionality is denoted by β and has dimensions of stiffness per unit length.

We define the relative compliance γ of the interface and reinforcement system by

$$\gamma = \min\left\{\frac{\beta^{-1}}{(2\mu_m)^{-1} - (2\mu_r)^{-1}},\ \frac{\beta^{-1}}{(2\kappa_m)^{-1} - (2\kappa_r)^{-1}}\right\}, \tag{6}$$

where γ has dimensions of length.

In this work we show that the interplay between the imperfect interface and the effect of the stiff reinforcement is mediated through a geometric quantity σ related to the shape and size of the reinforcement. We suppose the structure is reinforced by several domains containing stiff material. We focus our attention on one of the reinforcement domains Σ and examine its effect on the overall stiffness of the structure. The quantity σ associated with the reinforcement Σ is written as $\sigma(\Sigma)$. The quantity $\sigma(\Sigma)$ is the stationary value of the Rayleigh quotient:

$$\sigma(\Sigma) = \min_{\substack{\varphi \in \mathcal{C} \\ \varphi \neq 0}} \left\{ \frac{\int_{\partial\Sigma} \partial_s \varphi_{,i}\, \partial_s \varphi_{,i}\, dl}{\int_{\Sigma} \varphi_{,ij}\, \varphi_{,ij}\, dx} \right\}, \tag{7}$$

where,

$$\mathcal{C} = \left\{ \varphi \in H^{5/2}(\Sigma) \,|\, \Delta^2 \varphi = 0, \int_{\partial\Sigma} \varphi\, dl = 0, \int_{\partial\Sigma} \varphi_{,i}\, dl = 0, i = 1, 2. \right\}$$

Here ∂_s denotes tangential differentiation on the curve $\partial\Sigma$ and Δ^2 is the biharmonic operator. The notation $\varphi_{,i}$ represents differentiation, i.e., $\varphi_{,i} = \partial_{x_i} \varphi$ and $\varphi_{,ij} = \partial^2_{x_i x_j} \varphi$. The parameter σ has dimensions of inverse length. Conditions of stationarity for (7) deliver the eigenvalue problem

$$\Delta^2 \phi = 0, \text{ on } \Sigma, \tag{8}$$

$$n_i \partial_s^2 \phi_{,i} = -\sigma M_n(\phi), \text{ on } \partial\Sigma, \tag{9}$$

$$\partial_s(t_i \partial_s^2 \phi_{,i}) = -\sigma\{\partial_s M_s(\phi) + Q(\phi)\}, \text{ on } \partial\Sigma. \tag{10}$$

Here $M_n(\phi)$ is the bending moment of the reinforcement Σ given by $M_n(\phi) = n_i n_j \phi_{,ij}$ and $\partial_s M_s(\phi) + Q(\phi)$ is the Kirchoff shear force where $M_s(\phi) = t_i n_j \phi_{,ij}$ and $Q(\phi) = \partial_n \Delta\phi = 0$. We have denoted normal differentiation by ∂_n and the Laplace operator by Δ. From its definition we see that σ is the largest constant C for which the inequality

$$\int_{\partial\Sigma} \partial_s \varphi_{,i}\, \partial_s \varphi_{,i}\, dl \geq C \int_{\Sigma} \varphi_{,ij}\, \varphi_{,ij}\, dx. \tag{11}$$

holds for all φ in the space $\mathcal{C}$. For a disk of radius a one has that

$$\sigma = \frac{2}{3a}. \tag{12}$$

The overall stiffness of the reinforced structural cross section Ω is inversely proportional to the compliance energy of the structure associated

with a prescribed traction g. The criterion for when the imperfect interface overcomes the stiffening effect of the reinforcement domain Σ is given by

Compliance energy inequality.
If σ satisfies,

$$\gamma^{-1} \leq \sigma(\Sigma), \tag{13}$$

then for every load case g, the reinforcement does not reduce the compliance energy. *We emphasize that this result is independent of the location and geometry of the other components of the reinforcement phase and applies to every load case g.*

The compliance inequality naturally implies a reinforcement size effect. Here the length scale is set by γ. The effect of reinforcement size is seen clearly when Σ is a disk of radius a. We have the following size effect for a circular reinforcement.

Size effect for a circular reinforcement.
If the reinforcement is a disk of radius a, and

$$a \leq \frac{2}{3}\gamma, \tag{14}$$

then for every load case g, the reinforcement does not reduce the compliance energy.

This result gives a rigorous rule of thumb for the selection of the size of a reinforcement disk, namely: *Only disks of radius greater than $(2/3)\gamma$ can provide reinforcement. This statement holds true independently of where the disks are placed in the structure.* This result is in striking contrast to what is seen when there is perfect bonding between structural materials. For that situation, the addition of an infinitesimally small disk of stiffer material always reduces the compliance energy.

These results are directly applicable to problems of optimal compliance design for plane strain problems. A prototypical problem is the optimal design of a structure reinforced with disks of different radii. Each disk has compliance C_r^{-1} and the matrix has compliance C_m^{-1}. Here we suppose that each disk is made from stiffer material, i.e., $C_r^{-1} < C_m^{-1}$. The class of admissible designs is given by the set of all reinforcements consisting of a finite number of non-intersecting disks. We restrict the joint area of the disks to be less than a prescribed area fraction θ_r of the design domain Ω. However, no lower bound is placed on the size of the disks nor do we place a constraint on the number of disks appearing in any design. We show in this project that all energy minimizing configurations of disks can be found

among those that contain disks of radii greater than or equal to $\frac{2}{3}\gamma$ or no disks at all. It is evident that minimizing sequences made from progressively finer suspensions of disks (i.e., *homogenized designs*) can be excluded.

This work has appeared in

1. R. Lipton, Reinforcement of elastic structures in the presence of imperfect bonding, *Quarterly Journal of Applied Mathematics* (2001), pp. 353–364.

3 Work at Air Force laboratories.

I have been interacting with scientists in the Nonmetallic Materials Division at Wright Patterson Airforce Base since May of 2000. During the Summer of 2001 I met with scientists at the Air Force Research Laboratory at Wright Patterson Air Force Base and it was agreed that the design of functionally graded materials fits well with the research interests and goals of the Nonmetallic Materials and Ceramics Divisions. Much of the basic results on functionally graded materials obtained during the course of this project will provide the foundation for my interaction with the Air Force Laboratory. Future projects related to the design of composite sandwich structures and reinforced ceramics will be done in collaboration with Dr. Iarve of the Nonmetallic Materials Division and Dr. Kerans and Dr. Parthasarathy of the Ceramics Division. A third project motivated by results obtained under this grant will be to compute bounds on elastic field fluctuations for realizable classes of random geometries that have been developed by Dr. Jonathan Spowart of the Metallic Composites Division at the Air Force Research Laboratory.

4 Publications from grant sponsored activity given a Featured Review in Mathematical Reviews

Optimal fiber configurations for maximum torsional rigidity, *Archive for Rational Mechanics and Analysis*, 144 (1998), pp. 79–106.
Mathematical Reviews reference number: **2000i:74075.**

5 Publications resulting from grant sponsored activity.

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